PUSH-FORWARDS ON PROJECTIVE TOWERS

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Abstract. In this paper we derive a simple and useful combinatorial formula for the push-forwards of cohomology classes down projective towers, in terms of the push-forwards down the individual steps in the tower.

1. Introduction

1.1. Consider a proper map of algebraic varieties:

$$\pi : X' \longrightarrow X.$$

Pick any class $c \in A^*(X')$ and call it the **tautological class** of π. Relative to this choice, we can define the **Segre series** of π:

$$s(\pi, u) = \pi_* \left(\frac{1}{u - c} \right). \tag{1.1}$$

This series in u^{-1} has coefficients in $A^*(X)$, and it encodes the push-forwards of all powers of the tautological class c.

The terminology is motivated by the case when $X' = \mathbb{P}_X \mathcal{V}$ is the projectivization of a cone on X. In this case, we let $c = c_1(\mathcal{O}(1))$ and the above notion coincides (up to normalization) with the Segre class introduced by Fulton in [1]. In the particular case when $\mathcal{V}$ is a vector bundle, the Segre series equals the inverse of the (properly renormalized) Chern polynomial of $\mathcal{V}$.

1.2. The main subject of this paper are **projective towers**, namely compositions of proper maps of algebraic varieties:

$$\pi : X_k \xrightarrow{\pi^k} X_{k-1} \xrightarrow{\pi^{k-1}} ... \xrightarrow{\pi^2} X_1 \xrightarrow{\pi^1} X_0. \tag{1.2}$$

As before, pick $c_i \in A^*(X_i)$ and call them the **tautological classes** of the tower. We want to encode the push-forwards of these tautological classes under π, and the reasonable way to do this is to define the **Segre series** of the tower as:

$$s(\pi, u_1, ..., u_k) = \pi_* \left(\frac{1}{u_1 - c_1} \cdot ... \cdot \frac{1}{u_k - c_k} \right) \tag{1.3}$$

We suppress the obvious pull-back maps to X_k, and hope that this will cause no confusion.

1.3. One of the main technical results of [2] involves studying a particular projective tower (1.2). One needs to derive a closed formula for the Segre series of the whole tower π from the Segre series of the individual maps π^i. The assumption we make on these individual Segre series is that:

$$s(\pi^i, u) = \prod_{m_i} Q_{m_i}(u + m_i^{i-1} c_{i-1} + ... + m_i^1 c_1), \tag{1.4}$$

for some series Q_{m_i} with coefficients in $A^*(X_0)$, where the product goes over finitely many vectors $m_i = (m_i^1, ..., m_i^{i-1})$ of integers. This assumption will be motivated in section 2.1, based on the particular example of a tower of projective bundles. Then our main Theorem 2.3 below implies that:

$$s(\pi, u_1, ..., u_k) = \left[\prod_{i=1}^{k} \prod_{m_i} Q_{m_i}(u_i + m_i^{i-1} u_{i-1} + ... + m_i^1 u_1) \right]_- \tag{1.5}$$

The notation $[\dots]_-$ means that we expand each Q_{m_i} in non-negative powers of $u_{i-1}, ..., u_1$, and then we only keep the monomials with all negative exponents in the resulting formula.

1.4. The basic idea, naturally, is to successively push forward the tautological classes from X_k to X_{k-1} to $\dots$ to X_1 to X_0, and assumption (1.4) provides the means for this recursion. However, if one carried out this procedure straightforwardly, one would not obtain a closed formula. The reason why formula (1.5) looks so nice is that we are adding terms with non-negative exponents, only to get rid of them when we apply $[\dots]_-$ at the very end.

This closed formula is very useful in the papers [2] and [3]. In the present note, we will present a baby case of the main technical computation of these papers: we will rederive a closed formula for integrals on the complete flag variety of vector subspaces of a fixed vector space.

I would like to thank Mircea Mustata, Aaron Silberstein and Aleksander Tsymbaliuk for useful discussions.

2. Tautological Classes

2.1. Consider the special case of (1.2) where $X_i = \mathbb{P}_{X_{i-1}} \mathcal{V}_i$ for some vector bundle $\mathcal{V}_i$ of rank r_i on X_{i-1}, and $c_i = c_1(\mathcal{O}_i(1))$ is the first Chern class of the tautological line bundle. It is well-known ([1], Section 3.2) that the individual Segre classes are equal to the inverse Chern classes:

$$s(\pi^i, u) = c^{-1}(\mathcal{V}_i, u) \qquad \text{where} \qquad c(\mathcal{V}_i, u) = u^{r_i} \cdot \sum_{k=0}^{r_i} u^{-k} c_k(\mathcal{V}_i). \tag{2.1}$$

The above Chern classes only depend on the class of $\mathcal{V}_i$ in the $K-$theory of X_{i-1}. We will make the following assumption on this class:

$$[\mathcal{V}_i] = \sum_{m_i} [V_{m_i}] \otimes [\mathcal{O}_1(m_i^1)] \otimes ... \otimes [\mathcal{O}_{i-1}(m_i^{i-1})],$$

in $K-$theory, where the sum goes over finitely many vectors $m_i = (m_i^1, ..., m_i^{i-1})$ of integers and $[V_{m_i}] \in K(X_0)$ are arbitrary classes (we are suppressing the obvious pull-back maps, and hope that this will cause no confusion). In other words, we assume that in $K-$theory each $\mathcal{V}_i$ is constructed by twisting bundles on the lower steps in the tower by various tautological line bundles. Then the Whitney sum formula tells us that:

$$s(\mathcal{V}_i, u) = \prod_{m_i} s(V_{m_i}, u + m_i^{i-1} c_{i-1} + ... + m_i^1 c_1),$$

where the Segre series $s(V_{m_i}, u)$ now have coefficients in $A^*(X_0)$. This setup justifies our assumption (1.4).

2.2. Let us now go to a general projective tower (1.2) that satisfies assumption (1.4). Along with the variable u_i, for each $i \in \{1, ..., k\}$ pick an extra set of variables A_i. Then our main result is the following theorem:

Theorem 2.3. *We have the following relation:*

$$\pi_* \prod_{i=1}^{k} \left(\frac{1}{u_i - c_i} \prod_{u \in A_i} \frac{1}{u - c_i} \right) =$$

$$= \left[\prod_{i=1}^{k} \prod_{m_i} Q_{m_i}(u_i + m_i^{i-1} u_{i-1} + ... + m_i^1 u_1) \prod_{u \in A_i} \frac{1}{u - u_i} \right]_- \tag{2.2}$$

where we expand each Q_{m_i} in non-negative powers of $u_{i-1}, ..., u_1$, and each $(u - u_i)^{-1}$ in non-negative powers of u_i. The notation $[\dots]_-$ means that we only keep the terms for which all the u_i's and u's have negative exponents.

Relation (1.5) is simply the case when all the A_i are empty. Though the difference between (1.5) and (2.2) is a purely formal manipulation of series, we are working with this more general format for the purposes of [2].

Proof For each i between 0 and k, let us define the quantity:

$$Z_j = \pi_*^1 ... \pi_*^j \left[\prod_{i=1}^{j} \left(\frac{1}{u_i - c_i} \prod_{u \in A_i} \frac{1}{u - c_i} \right) \cdot \right.$$

$$\left. \cdot \prod_{i=j+1}^{k} \prod_{m_i} Q_{m_i}(u_i + m_i^{i-1} u_{i-1} + ... + m_i^{j+1} u_{j+1} + m_i^j c_j + ... + m_i^1 c_1) \prod_{u \in A_i} \frac{1}{u - u_i} \right]_-$$

It is easy to see that Z_k is the LHS and Z_0 is the RHS of (2.2). Therefore, to complete the proof of our theorem, we need to show that $Z_j = Z_{j-1}$, or in other words that:

$$\pi_*^j\left[\left(\frac{1}{u_j-c_j}\prod_{u\in A_j}\frac{1}{u-c_j}\right)\cdot\prod_{i=j+1}^{k}\prod_{m_i}Q_{m_i}(u_i+...+m_i^jc_j+...m_i^1c_1)\right]_-=$$

$$=\left[\prod_{u\in A_j}\frac{1}{u-u_j}\prod_{i=j}^{k}\prod_{m_i}Q_{m_i}(u_i+...+m_i^ju_j+...+m_i^1c_1)\right]_- \tag{2.3}$$

To prove this relation, it is enough to assume $Q_{m_i}(u)=u^{\alpha_{m_i}}$ and then the LHS becomes

$$\pi_*^j\left[\sum_{\beta_j,\beta_u,\beta_{m_i}}c_j^{\beta_j+\sum\beta_u+\sum\beta_{m_i}}u_j^{-\beta_j-1}\prod_{u\in A_j}u^{-\beta_u-1}\prod_{i>j}^{m_i}(m_i^j)^{\beta_{m_i}}(u_i+...)^{\alpha_{m_i}-\beta_{m_i}}\binom{\alpha_{m_i}}{\beta_{m_i}}\right]_-$$

where all the β's range over the non-negative integers, u ranges over A_j and i ranges over $\{j+1,...,k\}$. Now if we denote by γ the exponent of c_j and solve for β_j, the above becomes:

$$\pi_*^j\left[\sum_{\gamma,\beta_u,\beta_{m_i}}c_j^{\gamma}u_j^{-\gamma-1}\prod_{u\in A_j}\left(u_j^{\beta_u}u^{-\beta_u-1}\right)\prod_{i>j}^{m_i}(m_i^ju_j)^{\beta_{m_i}}(u_i+...)^{\alpha_{m_i}-\beta_{m_i}}\binom{\alpha_{m_i}}{\beta_{m_i}}\right]_-$$

The condition that $\beta_j\geq 0$, which was lost when we replaced it by the variable γ, is recovered by the condition $[\dots]_-$. Since γ and the β's sum independently, the above equals:

$$\left[\left(\pi_{j*}\frac{1}{u_j-c_j}\right)\prod_{u\in A_j}\frac{1}{u-u_j}\prod_{i=j+1}^{k}\prod_{m_i}Q_{m_i}(u_i+...+m_i^ju_j+...+m_i^1c_1)\right]_-$$

Then if we replace $\pi_{j*}(u_j-c_j)^{-1}$ by (1.4), the above yields the RHS of (2.3), thus completing the proof.

□

3. A Basic Example

Theorem (2.3) works equally well if we replace Chow rings by cohomology rings. For a simple example, let us consider the variety F of complete flags in $\mathbb{C}^{k+1}$:

$$V_1\subset...\subset V_k\subset\mathbb{C}^{k+1}, \tag{3.1}$$

where V_i is an $i-$dimensional subspace. On F, we will consider the universal vector bundle $\mathcal{V}_i$ whose fiber over (3.1) is V_i, and also the tautological line bundle:

$$\mathcal{O}_i(1)=\mathcal{V}_{k+2-i}/\mathcal{V}_{k+1-i}. \tag{3.2}$$

It is well known that $c_i = c_1(\mathcal{O}_i(1))$ as $i \in \{1, ..., k\}$ generate the cohomology of F. Then we have the following result:

Proposition 3.1. *The following identity tells us how to integrate any cohomology class on F:*

$$\int_F \frac{1}{(u_1 - c_1)...(u_k - c_k)} = (u_1...u_k)^{-k-1} \prod_{1 \leq i < j \leq k} (u_j - u_i).$$

Proof If we let F_i parametrize flags:

$$V_{k+1-i} \subset ... \subset V_k \subset \mathbb{C}^{k+1},$$

where each V_j still has dimension j, then $F_0 = \text{pt}$ and $F_k = F$. All these spaces fit into a projective tower:

$$\pi : F_k \xrightarrow{\pi^k} F_{k-1} \xrightarrow{\pi^{k-1}} ... \xrightarrow{\pi^2} F_1 \xrightarrow{\pi^1} F_0 = \text{pt}.$$

It is easy to see that $F_i = \mathbb{P}_{F_{i-1}}(\mathcal{V}^\vee_{k+2-i})$, so this realizes the flag variety as a tower of projective bundles over the point. It's easy to see that $\mathcal{O}_i(1)$ of this tower are precisely the line bundles (3.2), and therefore we have the following equality in the Grothendieck group of F_i:

$$[\mathcal{V}^\vee_{k+2-i}] = [\mathcal{O}^{k+1}] - [\mathcal{O}_1(-1)] - ... - [\mathcal{O}_{i-1}(-1)],$$

By the Whitney sum formula and (2.1), one therefore has:

$$s(\pi^i, u) = \frac{(u - c_1)...(u - c_{i-1})}{u^{k+1}},$$

Then (1.5) implies the desired result. We do not need the $[\dots]_-$ anymore, because all terms only consist of negative monomials already.

$\square$

References

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